Two Treatises on elliptic functions *

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1 First Treatise on elliptic Functions

1.1 A. On the Transformation of the Functions $E(u)$, $\Pi(u,a)$, which extend to the second and third kind of elliptic Integrals. On the Transformation of the Function $\Omega(u)$.

1.

Using the same notation I introduced in the *Fundamenta nova*, let n be an arbitrary odd number, let m, m' be arbitrary numbers, but both not divisible by the same factor of n at the same time: In the *Fundamenta nova* I proved a theorem, fundamental in the transformation theory of elliptic functions, that, having put $\omega = \frac{mK+m'iK'}{n}$,

$$\lambda = k^n \left\{\sin\operatorname{coam} 2\omega \sin\operatorname{coam} 4\omega \cdots \sin\operatorname{coam} 2(n-1)\omega\right\}^4,$$

$$M = (-1)^{\frac{n-1}{2}} \left\{\frac{\sin\operatorname{coam} 2\omega \sin\operatorname{coam} 4\omega \cdots \sin\operatorname{coam} 2(n-1)\omega}{\sin\operatorname{am} 2\omega \sin\operatorname{am} 4\omega \cdots \sin\operatorname{am}(n-1)\omega}\right\}$$

and additionally

$$x = \sin\operatorname{am} u, \qquad y = \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)$$

it will be

$$y = \frac{x}{M} \cdot \frac{\left(1 - \frac{x^2}{\sin^2\operatorname{am} 2\omega}\right)\left(1 - \frac{x^4}{\sin^2\operatorname{am} 2\omega}\right)\cdots\left(1 - \frac{x^2}{\sin^2\operatorname{am}(n-1)\omega}\right)}{(1 - k^2\sin^2\operatorname{am} 2\omega\cdot x^2)(1 - k^2\sin^2\operatorname{am} 4\omega\cdot x^2)\cdots(1 - k^2\sin^2\operatorname{am}(n-1)\omega\cdot x^2)}.$$

Further, from this formula we derived (*Fund.* § 23) an identical equation, which holds for arbitrary x:

$$(1.) \qquad x\prod(x^2 - \sin^2\operatorname{am} 2p\omega) - \frac{\lambda}{kM}\sin\left(\frac{u}{M}, \lambda\right)\prod\left(x^2 - \frac{1}{k^2\sin^2\operatorname{am} 2p\omega}\right)$$

$$= [x - \sin\operatorname{am} u][x - \sin(u+4\omega)]\cdots[x - \sin\operatorname{am}(4(n-1)\omega)],$$

if in the products, denoted by the prefixed sign $\prod$, all values 1, 2, 3, $\cdots$, $\frac{n-1}{2}$ are attributed to the number p.

Having compared the coefficients of the powers of the variable x on both sides of the equation to each other, formula (1.) gives us the sums of the combinations of the expressions

$$\sin\operatorname{am} u, \quad \sin\operatorname{am}(u+4\omega), \quad \sin\operatorname{am}(u+8\omega), \cdots, \quad \sin\operatorname{am}(u+4(n-1)\omega).$$

For the sake of an example, the sum of these expressions is

$$= \frac{\lambda}{kM}\sin\left(\frac{u}{M},\lambda\right);$$

the sum of *the products of two of them*

$$= -[\sin^2 2\omega + \sin^2 4\omega + \cdots + \sin^2\operatorname{am}(n-1)\omega],$$

which constant quantity we will denoted by $-\rho$. Hence one can also deduce the sum of the squares

$$\sin^2\operatorname{am} u + \sin^2\operatorname{am}(u+4\omega) + \cdots + \sin^2(u+4(n-1)\omega) = \frac{\lambda^2}{k^2M^2}\sin^2\operatorname{am}\left(\frac{u}{M},\lambda\right) + 2\rho,$$

or

$$(2.) \qquad \frac{\lambda^2}{k^2M^2}\sin^2\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sum \sin^2\operatorname{am} u - 2\rho,$$

if by $\sum\varphi(u)$ we denote the expression

$$\sum\varphi(p) = \varphi(u) + \varphi(u+4\omega) + \varphi(u+8\omega) + \cdots + \varphi(u+4(n-1)\omega).$$

From (2.) it also follows:

$$(3.) \qquad \frac{\lambda^2}{k^2M^2}\cos^2\operatorname{am}\left(\frac{u}{M},\lambda\right) = \Sigma\cos^2\operatorname{am} u - 2\sigma$$

$$(4.) \qquad \frac{1}{M^2}\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) = \Sigma\Delta\operatorname{am} u - 2\tau,$$

if

$$(5.) \qquad \frac{\lambda^2}{k^2M^2} = n - 2\rho - 2\sigma$$

$$(6.) \qquad \frac{1}{M^2} = n - 2k^2\rho + 2\tau.$$

Since the expression $\cos \operatorname{am}\left(\frac{u}{M}, \lambda\right)$, having put $u = K$, from (3.) we will obtain:

$$(7.) \quad \sigma = \cos^2 \operatorname{coam} 2\omega + \cos^2 \operatorname{coam} 4\omega + \cdots + \cos^2 \operatorname{coam}(n-1)\omega.$$

Since the expression $\Delta \operatorname{am}\left(\frac{u}{m}, \lambda\right)$, having put $u = K + iK'$, and additionally:

$$\Delta \operatorname{am}(u + K + iK') = \Delta \operatorname{coam}(u + iK') = ik' \tan \operatorname{am} u$$

(see *Fund.* § 19), from (4.) we will obtain:

$$(8.) \qquad \tau = k'k'[\tan^2 \operatorname{am} 2\omega + \tan^2 \operatorname{am} 4\omega + \cdots + \tan^2 \operatorname{am}(n-1)\omega].$$

2.

Formulas (2.), (3.), (4.) can also be represented this way:

$$(9.) \qquad \frac{\lambda^2}{k^2 M^2} \sin^2 \operatorname{am}\left(\frac{u}{M}, \lambda\right) = \sin^2 \operatorname{am} u + \Sigma[\sin^2 \operatorname{am}(u + 2p\omega) + \sin^2 \operatorname{am}(u - 2p\omega)] - 2\rho,$$

$$(10.) \qquad \frac{\lambda^2}{k^2 M^2} \cos^2 \operatorname{am}\left(\frac{u}{M}, \lambda\right) = \cos^2 \operatorname{am} u + \Sigma[\cos^2 \operatorname{am}(u + 2p\omega) + \cos^2 \operatorname{am}(u - 2p\omega)] - 2\sigma,$$

$$(11.) \qquad \frac{1}{M^2} \Delta^2 \operatorname{am}\left(\frac{u}{M}, \lambda\right) = \Delta^2 \operatorname{am} u + \Sigma[\Delta^2 \operatorname{am}(u + 2p\omega) + \Delta^2 \operatorname{am}(u - 2p\omega)] + 2\tau,$$

if always the values 1, 2, 3, $\cdots$, $\frac{n-1}{2}$ are attributed to the number p.
Now put:

$$\int_0^u \Delta^2 \operatorname{am} u du = E(u),$$

which deviates slightly from the notation used by Legendre, which we also used partly in the *Fundamenta nova*. For, having put $\varphi = \operatorname{am} u$, he denoted the elliptic integrals of the second kind by:

$$E(\varphi) = E(\operatorname{am} u) = \int_0^\varphi \Delta\varphi d\varphi = \int_0^u \Delta^2 \operatorname{am} u du$$

so that what is $E(u)$ for us, is $E(\operatorname{am} u)$ for him. Further, by the letter E, without an argument, we always denote the function:

$$E = E(K) = \int_0^{\frac{\pi}{2}} \Delta\varphi d\varphi,$$

which he denotes by E^{I}, and in the same way by E' the function:

$$E' = E(K', k') = \int_0^{\frac{\pi}{2}} \Delta(\varphi, k') d\varphi.$$

Having constituted this, we have:

$$\int_0^u [\Delta^2 \operatorname{am}(u + 2p\omega) + \Delta^2 \operatorname{am}(u - 2p\omega)] du = E(u + 2p\omega) + E(u - 2p\omega).$$

One will not have to add a constant, since both sides of the equation vanish for $u = 0$. Hence from (11.), having integrated from the limit $u = 0$ to $u = u$:

$$(12.) \qquad \frac{1}{M} E\left(\frac{u}{M}, \lambda\right) = E(u) + \sum [E(u + 2p\omega) + E(u - 2p\omega)] + 2\tau u.$$

Formula (12.) can be transformed by means of the known theorem on the addition of elliptic integrals of the second kind:

$$E(u + a) - E(u - a) = 2E(u) - \frac{2k^2 \sin^2 \operatorname{am} a \sin \operatorname{am} u \cos \operatorname{am} u \Delta amu}{1 - k^2 \sin^2 a \sin^2 \operatorname{am} u}$$

which, after a differentiation, is easily demonstrated from the elements (see. *Fund.* § 49). By means of this (12.) goes over into this one:

$$(13.) \quad nE(u) - \frac{1}{M} E\left(\frac{u}{M}, \lambda\right) + 2\tau u = 2k^2 \sin \operatorname{am} u \cos \operatorname{am} u \Delta \operatorname{am} u \sum \frac{\sin^2 \operatorname{am} 2p\omega}{1 - k^2 \sin^2 \operatorname{am} 2p\omega \sin^2 \operatorname{am} u}.$$

Formulas (12.), (13.) concern the transformation of elliptic integrals of the second kind. We will exhibit them in a more convenient form soon.
Let us put:

$$\int_0^u E(u)du = \log\Omega(u),$$

since

$$\frac{2k^2 \sin\operatorname{am} u \cos\operatorname{am} u \Delta\operatorname{am} u \sin^2 am2p\omega}{1 - k^2 \sin^2 2p\omega \sin^2 \operatorname{am} u} = -\frac{d\log(1 - k^2 \sin^2 \operatorname{am} 2p\omega \sin^2 \operatorname{am} u)}{du},$$

from (13.), again after the integration from $u = 0$ to $u = u$, we obtain:

$$n\log\Omega(u) - \log\Omega\left(\frac{u}{M}, \lambda\right) + \tau uu = -\sum \log(1 - k^2 \sin^2 \operatorname{am} 2p\omega \sin^2 \operatorname{am} u),$$

or:

$$(14.) \qquad e^{-\tau uu} \cdot \frac{\Omega\left(\frac{u}{M}, \lambda\right)}{\Omega^n(u)} = \prod(1 - k^2 \sin^2 \operatorname{am} 2p\omega \sin^2 \operatorname{am} u),$$

if, as above, by $\prod \varphi(p)$ one denotes the product

$$\prod \varphi(p) = \varphi(1)\varphi(2)\varphi(3)\cdots\varphi\left(\frac{n-1}{2}\right).$$

Having put $\sin\operatorname{am} u = x$ (14.) is represented this way:

$$(15.) \quad e^{-\tau uu} \cdot \frac{\Omega\left(\frac{u}{M}, \lambda\right)}{\Omega^n(u)} = (1 - k^2 \sin^2 \operatorname{am} 2\omega \cdot xx)(1 - k^2 \sin^2 \operatorname{am} 4\omega \cdot xx)\cdots(1 - k^2 \sin^2 \operatorname{am}(n-1)\omega \cdot xx).$$

This expression constitutes the denominator of the rational substitution that was used for the transformation of the elliptic functions (see above),

$$\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right) = \frac{\frac{x}{M}\left(1 - \frac{xx}{\sin^2 \operatorname{am} 2\omega}\right)\left(1 - \frac{xx}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1 - \frac{xx}{\sin^2 \operatorname{am}(n-1)\omega}\right)}{(1 - k^2 \sin^2 \operatorname{am} 2\omega \cdot xx)(1 - k^2 \sin^2 \operatorname{am} 4\omega \cdot xx)\cdots(1 - k^2 \sin^2 \operatorname{am}(n-1)\omega \cdot xx)};$$

therefore, it is possible to express this denominator by means of the new transcendent $\Omega(u)$ separately. This is a most important theorem and of highest use in the whole theory of elliptic functions.
Let that substitution, if $x = \sin\operatorname{am} u$, be

$$\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right) = \frac{x}{M} \cdot \frac{1 + A'x^2 + A''x^4 + \cdots + A^{\left(\frac{n-1}{2}\right)}x^{n-1}}{1 + B'x^2 + B''x^4 + \cdots + B^{\left(\frac{n-1}{2}\right)}x^{n-1}},$$

having put

$$1 + A'x^2 + A''x^4 + \cdots + A^{\left(\frac{n-1}{2}\right)}x^{n-1} = \prod\left(1 - \frac{\sin^2 \operatorname{am} u}{\sin^2 \operatorname{am} 2p\omega}\right)$$

$$= \left(1 - \frac{\sin^2 \operatorname{am} u}{\sin^2 \operatorname{am} 2\omega}\right)\left(1 - \frac{\sin^2 \operatorname{am} u}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1 - \frac{\sin^2 \operatorname{am} u}{\sin^2 \operatorname{am}(n-1)\omega}\right),$$

$$1 + B'x^2 + B''x^4 + \cdots + B^{\left(\frac{n-1}{2}\right)}x^{n-1} = \prod(1 - k^2 \sin^2 \operatorname{am} 2p\omega \sin^2 \operatorname{am} u)$$

$$= (1 - k^2 \sin^2 \operatorname{am} 2\omega \sin^2 \operatorname{am} u)(1 - k^2 \sin^2 \operatorname{am} 4\omega \sin^2 \operatorname{am} u)\cdots(1 - k^2 \sin^2 \operatorname{am}(n-1)\omega \sin^2 \operatorname{am} u),$$

it will be:

$$(16.) \qquad e^{-\tau uu}\frac{\Omega\left(\frac{u}{M},\lambda\right)}{\Omega^n(u)} = 1 + B' \sin^2 \operatorname{am} u + B'' \sin^4 \operatorname{am} u + \cdots + B^{\left(\frac{n-1}{2}\right)} \sin^{n-1} \operatorname{am} u.$$

Hence after a logarithmic differentiation it follows

$$(17.) \qquad nE(u) - \frac{1}{M}E\left(\frac{u}{M},\lambda\right) + 2\tau u$$

$$= -\frac{\cos \operatorname{am} u \Delta u[2B' \sin \operatorname{am} u + 4B'' \sin^3 \operatorname{am} u + \cdots + (n-1)B^{\left(\frac{n-1}{2}\right)} \sin^{n-2} \operatorname{am} u]}{1 + B' \sin^2 \operatorname{am} u + B'' \sin^4 \operatorname{am} u + \cdots + B^{\left(\frac{n-1}{2}\right)} \sin^{n-1} \operatorname{am} u}.$$

This most elegant formula tells us, how from the denominator of the expression found for the transformed function $\sin \operatorname{am}\left(\frac{u}{M},\lambda\right)$ one always finds a transformation of an elliptic integral of the second kind.
You will find the value of the constant τ by putting u infinitely small, having done which we have $E(u) = u$, $\sin \operatorname{am} u = u$, $\cos \operatorname{am} u = \Delta \operatorname{am} u = 1$, whence

$$n - \frac{1}{MM} + 2\tau = -2B',$$

what, since $B' = -k^2\rho$, agrees with formula (6.).
Additionally, let us note, if you start from formula (12.), that after the integration you obtain:

$$(18.) \qquad \Omega\left(\frac{u}{M},\lambda\right) = e^{\tau uu} \cdot \left\{\begin{array}{c} \Omega(u)\dfrac{\Omega(u+2\omega)\Omega(u+4\omega)\cdots\Omega(u+(n-1)\omega)}{\Omega(2\omega)\Omega(4\omega)\cdots\Omega((n-1)\omega)} \\ \times\dfrac{\Omega(u-2\omega)\Omega(u-4\omega)\cdots\Omega(u-(n-1)\omega)}{\Omega(2\omega)\Omega(4\omega)\cdots\Omega((n-1)\omega} \end{array}\right\}$$

3.

For the sake of brevity, if $x = \sin\operatorname{am} u$, let us put

$$U = \frac{x}{M}\left(1 + A'x^2 + A''x^4 \cdots + A^{\left(\frac{n-1}{2}\right)}x^{n-1}\right)$$
$$V = \quad\left(1 + B'x^2 + B''x^4 \cdots + B^{\left(\frac{n-1}{2}\right)}x^{n-1}\right).$$

so that:

$$\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right) = \frac{U}{V}.$$

(17.) becomes:

$$nE(u) - \frac{1}{M}E\left(\frac{u}{M}, \lambda\right) + 2\tau u = -\frac{dV}{Vdu}.$$

whence, after a differentiation,

$$n\Delta^2\operatorname{am} u - \frac{1}{M^2}\Delta^2\operatorname{am}\left(\frac{u}{M}, \lambda\right) + 2\tau = \frac{dvdV - Vd^2V}{VVdu^2},$$

which formula, recalling (6.):

$$\frac{1}{M^2} = n - 2k^2\rho + 2\tau,$$

goes over into this one:

$$\text{(19.)} \qquad -nk^2\sin^2\operatorname{am} u + \frac{\lambda^2}{M^2}\left(\frac{u}{M}, \lambda\right) + 2k^2\rho = \frac{dVdV - Vd^2V}{VVdu^2},$$

or, after a multiplication by VV, into this one:

$$k^2(2\rho - n\sin^2\sin\operatorname{am} u)VV + \frac{\lambda^2}{M^2}UU = \frac{dV}{du}\frac{dV}{du} - V\frac{d^2V}{du^2}.$$

Further, we saw in the *Fundamenta nova* § 21, having put $u + iK'$ instead of u or $\frac{1}{k\sin\operatorname{am} u}$ instead of $\sin\operatorname{am} u$, that

$$V \quad \text{goes over into} \quad \sqrt{\frac{\lambda}{k^n}}\cdot\frac{U}{\sin^n\operatorname{am} u}, \qquad \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right) \quad \text{into} \quad \frac{1}{\lambda\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)},$$

whence the expression:

$$\frac{dVdV - Vd^2V}{VVdu^2} = -\frac{d^2 \log V}{du^2}$$

goes over into:

$$\frac{nd^2 \log \sin \operatorname{am} u}{du^2} - \frac{d^2 \log U}{du^2} = n\left\{k^2 \sin^2 \operatorname{am} u - \frac{1}{\sin^2 \operatorname{am} u}\right\} - \frac{d^2 \log U}{du^2},$$

and hence (19.) into:

$$-nk^2 \sin^2 \operatorname{am} u + \frac{1}{M^2 \sin^2 \operatorname{am}\left(\frac{u}{M}, \lambda\right)} + 2k^2\rho = \frac{dUdU - Ud^2U}{UUdu^2},$$

whence, after a multiplication by UU:

$$k^2(2\rho - n \sin^2 \operatorname{am} u)VV + \frac{1}{M^2}VV = \frac{dU}{du}\frac{dU}{du} - U\frac{d^2U}{du^2}.$$

To the formulas we found:

$$\text{(20.)} \qquad k^2(2\rho - n \sin^2 \operatorname{am} u)VV - \frac{\lambda^2}{M^2}UU = \frac{dV}{du}\frac{dV}{du} - V\frac{d^2V}{du^2},$$

$$\text{(21.)} \qquad k^2(2\rho - n \sin^2 \operatorname{am} u)UU - \frac{1}{M^2}VV = \frac{dU}{du}\frac{dU}{du} - U\frac{d^2U}{du^2}$$

one has to add this one:

$$\text{(22.)} \qquad V\frac{dU}{du} - U\frac{dV}{du} = \frac{1}{M}\sqrt{(VV - UU)(VV - \lambda^2 UU)},$$

which results from the differentiation of the equation $\sin \operatorname{am}\left(\frac{u}{M}, \lambda\right) = \frac{U}{V}$; by means of it it is possible to eliminate one of the quantities U, V from (20.), (21.); having done so, one will get to a differential equation of third order. This is indeed a memorable and very deep theorem, THAT THE NUMERATOR AND THE DENOMINATOR OF THE SUBSTITUTION, U, V, CAN BOTH BE DEFINED BY AN DIFFERENTIAL EQUATION OF THIRD ORDER.

For the sake of brevity I will not state these differential equations of third order here; in all cases it seems to be more convenient, to use the equations (20.) − (22.) in combination instead of the third order equations. I will demonstrate their extraordinary use for the algebraic formation of the functions U, V, or

the substitution, which leads to the transformation, on another occasion. Here, let us only note the following proof for the formulas (20.), (21.).
For, having divided (20.) by VV, (21.) by UU, it results:

$$\begin{aligned} k^2(2\rho - n\sin^2 \operatorname{am} u) + \frac{\lambda^2}{M^2}\sin^2 \operatorname{am}\left(\tfrac{u}{M}, \lambda\right) &= -\frac{d^2 \log V}{du^2}, \\ k^2(2\rho - n\sin^2 \operatorname{am} u) + \frac{1}{M^2 \sin^2 \operatorname{am}\left(\tfrac{u}{M}, \lambda\right)} &= -\frac{d^2 \log U}{du^2}, \end{aligned}$$

whence, after a subtraction,

$$\frac{1}{M^2}\left\{\lambda^2 \sin^2 \operatorname{am}\left(\frac{u}{M}, \lambda\right)\right\} - \frac{1}{\sin^2 \operatorname{am}\left(\frac{u}{M}, \lambda\right)} = \frac{d^2 \log \sin \operatorname{am}\left(\frac{u}{M}, \lambda\right)}{du^2},$$

which results immediately from the formula:

$$\frac{d^2 \log \sin \operatorname{am} u}{du^2} = k^2 \sin^2 \operatorname{am} u - \frac{1}{\sin^2 \operatorname{am} u},$$

having put $\frac{u}{M}$ instead of u and λ instead of k.
The complete integral of the differential equations of third order, by which the functions U, V, are defined, do not seem to be accessible.

4.

Having integrated the formula mentioned above in § 2:

$$(23.) \quad E(u+a) + E(u-a) = 2E(u) - \frac{2k^2 \sin^2 \operatorname{am} a \sin \operatorname{am} u \cos \operatorname{am} u \Delta \operatorname{am} u}{1 - k^2 \sin^2 a \sin^2 \operatorname{am} u}$$

from $u = 0$ to $u = u$, we obtain:

$$\log \frac{\Omega(u+a)}{\Omega(a)} + \log \frac{\Omega(u-a)}{\Omega(a)} = 2\log \Omega(u) + \log(1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u),$$

whence a formula, fundamental in the analysis of the function Ω, results:

$$(24.) \quad \frac{\Omega(u+a)\Omega(u-a)}{\Omega^2(a)\Omega^2(u)} = 1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u.$$

From formula (23.), having commuted a and u, we find:

$$(25.) \qquad E(u+a) - E(u-a) = 2E(a) - \frac{2k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} \sin^2 \operatorname{am} u}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 u},$$

having integrated which from $u = 0$, we obtain:

$$\log \frac{\Omega(u+a)}{\Omega(u-a)} - 2uE(a) = -2k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \int_0^u \frac{\sin^2 \operatorname{am} u du}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u}.$$

In the *Fundamenta nova* I used the character $\Pi(u, a)$ to denote the integral, which according to Legendre's classification is an elliptic of the third kind:

$$\Pi(u, a) = k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta a \int_0^u \frac{\sin^2 \operatorname{am} u du}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u},$$

using which notation we have:

$$(26.) \qquad \Pi(u, a) = uE(a) + \frac{1}{2} \log \frac{\Omega(u-a)}{\Omega(u+a)}.$$

This is the fundamental formula for the reduction of elliptic integrals of the third kind to the functions $E(u)$, $\Omega(u)$. Confer *Fund.* §§ 49 and the following. By means of formula (26.) from the formulas found for the transformation of the functions $E(u)$, $\Omega(u)$ we immediately obtain those for the elliptic integrals of the third kind or of the function Π. For, from (26.), having put $\frac{u}{M}$, $\frac{a}{M}$, λ instead of u, a, k:

$$(27.) \qquad \Pi\left(\frac{u}{M}, \frac{a}{M}, \lambda\right) = \frac{u}{M} E\left(\frac{a}{M}, \lambda\right) + \frac{1}{2} \log \frac{\Omega\left(\frac{u-a}{M}, \lambda\right)}{\Omega\left(\frac{u+a}{M}, \lambda\right)},$$

from which formula we want to subtract the following:

$$n\Pi(u, a) = nE(a) + \frac{n}{2} \log \frac{\Omega(u-a)}{\Omega(u+a)},$$

it results:

$$(28.) \qquad \Pi\left(\frac{u}{M}, \frac{a}{M}, \lambda\right) - n\Pi(u, a)$$

$$= u\left\{\frac{1}{M}E\left(\frac{a}{M},\lambda\right) - nE(a)\right\} + \frac{1}{2}\log\frac{\Omega\left(\frac{u-a}{M},\lambda\right)}{\Omega^n(u-a)} - \frac{1}{2}\log\frac{\Omega\left(\frac{u+a}{M},\lambda\right)}{\Omega^n(u+a)},$$

which formula, using (16.), (17.) goes over in the following:

$$(29.)\qquad \Pi\left(\frac{u}{M},\frac{a}{M},\lambda\right) - n\Pi(u,a)$$

$$= \left\{\frac{\cos\operatorname{am} a\Delta\operatorname{am} a[2B'\sin\operatorname{am} a + 4B''\sin^3\operatorname{am} a + \cdots + (n-1)B^{\left(\frac{n-1}{2}\right)}\sin^{n-2}\operatorname{am} a]}{1 + B'\sin^2\operatorname{am} a + B''\sin^4\operatorname{am} a + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am} a}\right\} u$$

$$+\frac{1}{2}\log\frac{1 + B'\sin^2\operatorname{am}(u-a) + B''\sin^4\operatorname{am}(u-a) + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}(u-a)}{1 + B'\sin^2\operatorname{am}(u+a) + B''\sin^4\operatorname{am}(u+a) + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}(u+a)},$$

which fundamental formula teaches, how from the denominator of the substitution one immediately finds the transformation of elliptic integrals of the third kind.

The same can be exhibited in another way by the formulas (12.), (18.), by means of which

$$\Pi\left(\frac{u}{M},\frac{a}{M},\lambda\right) - \frac{u}{M}E\left(\frac{a}{M},\lambda\right) + 2\tau a u$$

$$= \frac{1}{2}\log\frac{\Omega(u-a)}{\Omega(u+a)} + \sum\left\{\frac{1}{2}\log\frac{\Omega(u+2p\omega-a)}{\Omega(u+2p\omega+a) + \frac{1}{2}\log\frac{\Omega(u-2p\omega-a)}{\Omega(u-2p\omega+a)}}\right\}$$

$$= \frac{1}{2}\log\frac{\Omega(u-a)}{\Omega(u+a)} + \sum\left\{\frac{1}{2}\log\frac{\Omega(u-a+2p\omega)}{\Omega(u+a-2p\omega) + \frac{1}{2}\log\frac{\Omega(u-a-2p\omega)}{\Omega(u+a+2p\omega)}}\right\},$$

whence we deduce the following two formulas:

$$(30.)\qquad \Pi\left(\frac{u}{M},\frac{a}{M},\lambda\right) + u\left\{nE(a) - \frac{1}{M}E\left(\frac{a}{M},\lambda\right) + 2\tau a\right\}$$

$$= \Pi(u,a) + \Pi(u+2\omega,a) + \Pi(u+4\omega,a) + \cdots + \Pi(u+(n-1)\omega,a)$$

$$+ \Pi(u-2\omega,a) + \Pi(u-4\omega,a) + \cdots + \Pi(u-(n-1)\omega,a)$$

$$(31.) \qquad \Pi\left(\tfrac{u}{M}, \tfrac{a}{M}, \lambda\right) = \Pi(u,a) + \Pi(u+2\omega,a) + \Pi(u+4\omega,a) + \cdots + \Pi(u+(n-1)\omega,a)$$
$$+ \Pi(u-2\omega,a) + \Pi(u-4\omega,a) + \cdots + \Pi(u-(n-1)\omega,a);$$

these are the new fundamental formulas. In the *Fund.* § 55 (7.) we gave the formula:

$$\Pi(u, a+b) + \Pi(u, a-b) - 2\Pi(u,a)$$
$$= -2k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \frac{\sin^2 \operatorname{am} b}{1 - k^2 \sin^2 \operatorname{am} b \sin^2 \operatorname{am} a} \cdot u + \frac{1}{2} \log \frac{1 - k^2 \sin^2 \operatorname{am} b \sin^2 \operatorname{am}(u-a)}{1 - k^2 \sin^2 \operatorname{am} b \sin^2 (u+a)},$$

by means of which from (31.):

$$(32.) \qquad \Pi\left(\frac{u}{M}, \frac{a}{M}, \lambda\right) - n\Pi(u,a)$$
$$= -2k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \sum \frac{\sin^2 am 2p\omega}{1 - k^2 \sin^2 \operatorname{am} 2p\omega \sin^2 \operatorname{am} a} \cdot u + \sum \frac{1}{2} \log \frac{1 - k^2 \sin^2 \operatorname{am} 2p\omega \sin^2 \operatorname{am}(u-a)}{1 - k^2 \sin^2 2p\omega \sin^2 \operatorname{am}(u+a)},$$

if you attribute the values 1, 2, 3, $\cdots$, $\frac{n-1}{2}$ to the number p. This formula also follows easily from (29.).

1.2 B. On the simply periodic Functions $\chi(u) = e^{\tau uu}\Omega(u)$ and their singular Properties.

5.

Let us examine our function $\Omega(u)$ more accurately, and let us give its reduction for an imaginary argument of the form iu to a real argument first.
Having put $\sin\varphi = i \tan\psi$:

$$\frac{d\varphi}{\Delta\varphi} = \frac{id\psi}{\Delta(\psi, k')}$$
$$\Delta\varphi d\varphi = \frac{i\Delta(\psi,k')d\psi}{\cos^2\psi},$$

whence, after an integration:

$$\int_0^\varphi \Delta\varphi d\varphi = i\left\{\tan\psi\Delta(\psi,k') + \int_0^\psi \frac{k'k'\sin^2\psi}{\Delta(\psi,k')} d\varphi\right\}.$$

This formula, having put:

$$\varphi = \mathrm{am}(iu, k'), \quad \text{whence} \quad \psi = \mathrm{am}(u, k'),$$

is represented this way in our notation:

$$(1.) \qquad E(iu) = i[\tan \mathrm{am}(u, k')\Delta \,\mathrm{am}(u, k') + u - E(u, k')],$$

whence, after an integration:

$$\log \Omega(iu) = \log \cos \mathrm{am}(u, k') - \frac{uu}{2} + \log \Omega(u.k'),$$

or:

$$(2.) \qquad \Omega(iu) = e^{-\frac{uu}{2}} \cos \mathrm{am}(u, k')\Omega(u, k').$$

Confer *Fund.* § 56 (1.), (2.).

6.

Having mentioned these things in advance, let us now ask, which changes the function $\Omega(u)$ undergoes, while the elliptical functions remain unchanged, i.e., while the argument u is changed into $u + 4mK + 4M'iK'$ for positive or negative numbers m, m'.
From the elements we know the formula:

$$(3.) \qquad E(u + 2mK) = E(u) + 2mE,$$

if by the simple letter E without the argument we denote the complete function $E(K)$ that Legendre denoted by E^{I}; having added a slash, by the character E' we will denote the complete function which extends to the complement of the modulus or the function $E' = E(K', k')$, as we indicated at the beginning. After an integration of (3.), we obtain:

$$\log \frac{\Omega(u + 2mK)}{\Omega(2mK)} = 2mE \cdot u + \log \Omega(u),$$

or:

$$(4.) \qquad \frac{\Omega(u + 2mK)}{\Omega(2mK)} = e^{2mE \cdot u}\Omega(u).$$

Having put $u = -2mK$ in this formula, since $\Omega(-u) = \Omega(u)$, $\Omega(0) = 1$, it results:

$$\Omega(2mK) = e^{2mmEK},$$

by means of which (4.) goes over into:

$$(5.) \qquad \Omega(u + 2mK) = e^{2mE\cdot(u+mK)}\Omega(u),$$

or into:

$$(6.) \qquad e^{-\frac{E}{2K}(u+2mK)^2}\Omega(u + 2mK) = e^{-\frac{Euu}{2K}}\Omega(u),$$

which formula teaches THAT THE FUNCTION

$$e^{-\frac{Euu}{2K}}\Omega(u),$$

HAVING CHANGED u INTO $u + 2mK$, REMAINS UNCHANGED AND HENCE HAS A REAL PERIOD WITH THE ELLIPTIC FUNCTIONS OF THE ARGUMENT u IN COMMON. Put $u + 2M'K'$ instead u in formula (2.), we find:

$$\Omega(iu + 2m'iK') = (-1)^{m'}e^{-\frac{(u+2m'K')^2}{2}}\cos\mathrm{am}(u,k')\Omega(u + 2m'K',k'),$$

whence, since from (6.):

$$e^{-\frac{E'}{2K'}(u+2m'K')^2}\Omega(u + 2m'K',k') = e^{-\frac{E'uu}{2K'}}\Omega(u,k'),$$

we obtain:

$$e^{-\frac{E'}{2K'}(u+2m'K')^2}\Omega(iu + 2m'iK') = (-1)^{m'}e^{-\frac{(u+2m'K')^2}{2}}\cos\mathrm{am}(u,k')e^{-\frac{E'uu}{2K'}}\Omega(u,k'),$$

or:

$$e^{\frac{K'-E'}{2K'}(u+2m'K')^2}\Omega(iu + 2m'iK') = (-1)^{m'}e^{-\frac{E'uu}{2K}}\cos(u,k')\Omega(u,k')$$

$$(-1)^{m'}e^{\frac{(K'-E)uu}{2K'}}\Omega(iu),$$

whence, having put $-iu$ instead u, or u instead of iu:

$$(7.) \qquad e^{-\frac{K'-E'}{2K'}(u+2m'iK')^2}\Omega(u + 2m'iK') = (-1)^{m'}e^{-\frac{K'-E'}{2K'}uu}\Omega(u),$$

which formula tells us THAT THE EXPRESSION

$$e^{-\frac{(K'-E')uu}{2K'}}\Omega(u),$$

HAVING CHANGED u INTO $u+4m'iK'$, REMAINS UNCHANGED, OR HAS ANOTHER IMAGINARY PERIOD WITH THE ELLIPTIC FUNCTIONS OF THE ARGUMENT u IN COMMON.

It should be noted that from the known formula, found by Legendre,

$$KE'+K'E-KK'=\frac{\pi}{2}$$

or:

$$\frac{E}{K}+\frac{E'}{K'}-1=\frac{\pi}{2KK'}.$$

it follows:

$$-\frac{K'-E'}{2K'}=\frac{\pi}{4KK'}-\frac{E}{2K},$$

whence formula (7.) can also represented this way:

$$(8.)\qquad e^{\left(\frac{\pi}{4KK'}-\frac{E}{2K}\right)(u+2m'iK')^2}\Omega(u+2m'iK')=(-1)^{m'}e^{\left(\frac{\pi}{4KK'}-\frac{E}{2K}\right)}\Omega(u).$$

Having changed u to $u+2mK$ in this formula, from (6.) it results:

$$e^{\left(\frac{\pi}{4KK'}-\frac{E}{2K}\right)(u+2mK+2m'iK')^2}\Omega(u+2mK+2m'iK')=(-1)^{m'}e^{\frac{\pi}{4KK'}(u+2mK)^2}e^{-\frac{Euu}{2K}}\Omega(u).$$

But:

$$\begin{aligned}&\frac{\pi}{4KK'}[(u+2mK+2m'iK')^2-(u+2mK)^2]\\ =&\frac{m'i\pi}{4K}[4u+4mK+4m'iK']+mm'i\pi\\ =&\frac{m'i\pi}{4K(mK+m'iK')}[(u+2mK+2m'iK)^2-uu]+mm'i\pi.\end{aligned}$$

Hence, after we note that $e^{mm'i\pi}=(-1)^{mm'}$ and, for the sake of brevity, put:

$$r=\frac{m'i\pi}{4K(mK+m'iK')}-\frac{E}{2K},$$

we obtain the formula:

$$(9.) \qquad e^{r(u+2mK+2m'iK')^2}\Omega(u+2mK+2m'iK') = (-1)^{m'(m+1)}e^{ruu}\Omega(u),$$

which formula teaches THAT THE

$$e^{\left(\frac{m'i\pi}{4K(mK+m'iK')}\right)uu}\Omega(u) = e^{ruu}\Omega(u),$$

HAVING CHANGED u INTO $u+4mK+4m'iK'$, REMAINS UNCHANGED, WHENCE IT ALSO HAS A PERIOD WITH THE ELLIPTIC FUNCTIONS OF THE ARGUMENT u IN COMMON.

It is convenient to note that the value of r is not changed, if one puts pm, pm' instead of m, m'.

Formula (9.) can also be represented this way:

$$\begin{aligned}(10.) \qquad \Omega(u+2mK+2m'iK') &= (-1)^{m'(m+1)}e^{-4r(mK+m'iK')(u+mK+m'iK')}\Omega(u)\\ &= (-1)^{m'(m+1)}e^{\frac{2K(mK+m'iK')}{K}(u+mK+m'iK')}\Omega(u),\end{aligned}$$

which general formula teaches, which changes the function $\Omega(u)$ undergoes, while the elliptic functions remain unchanged. Having put $u=0$, we obtain from (10.):

$$(11.) \qquad \Omega(2mK+2m'iK') = (-1)^{m'}e^{\frac{2E}{K}(mK+m'iK')^2+m'm'\pi\frac{K'}{K}}.$$

After a logarithmic differentiation we obtain from (10.):

$$\begin{aligned}(12.) \qquad E(u+2mK+2m'iK') &= E(u)+\frac{2E(mK+m'iK')}{K}-\frac{m'i\pi}{K}\\ &= E(u)+2ME+2m'i(K'-E'),\end{aligned}$$

whence, having put $u=0$,

$$(13.) \qquad E(2mK+2m'iK') = 2mE+2m'i(K'-E').$$

7.

In the following, let us put:

$$\chi(u) = e^{\tau uu}\Omega(u),$$

from (9.), having put $2m$, $2m'$ instead m, m', it will be:

$$\chi(u + 4mK + 4m'iK') = \chi(u)$$

so that $\chi(u)$ is a periodic function. Therefore, for innumerable values r can take on, while you attribute the one or the other to the numbers m, m', we obtained innumerable periodic functions $\chi(u)$, which have one single period with the elliptic functions in common. And vice versa, whatever period from the innumerable ones the elliptic functions have you want to choose, the quantity r can always be determined in such a way that the function:

$$\chi(u) = e^{ruu}\Omega(u)$$

enjoys the same period. From those various periodic functions $\chi(u)$ we chose that one in the *Fundamenta nova* which has a real period with the elliptic functions in common, for which $m' = 0$ and hence $r = -\frac{E}{2K}$. There we denoted that function by the particular character Θ so that:

$$\frac{\theta(u)}{\theta(0)} = e^{-\frac{Euu}{2K}}\Omega(u),$$

and everything what was propounded on the function Θ there, is either directly or very easily extended to the more general function $\chi(u)$.
From the formulas exhibited above:

$$\frac{\Omega(u+a)\Omega(u-a)}{\Omega^2(a)\Omega^2(u)} = 1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u,$$

$$\Pi(u,a) = \frac{u}{\Omega(a)} \cdot \frac{d\Omega(a)}{da} + \tfrac{1}{2}\log\frac{\Omega(u-a)}{\Omega(u+a)},$$

it also follows, having put:

$$\chi(u) = e^{ruu}\Omega(u),$$

whatever constant r is:

$$\text{(14.)} \qquad \frac{\chi(u+a)\chi(u-a)}{\chi^2(a)\chi^2(u)} = 1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u,$$

$$\text{(15.)} \qquad \Pi(u,a) = \frac{u}{\chi(a)} \cdot \frac{d\chi(a)}{da} + \tfrac{1}{2} \log \frac{\chi(u-a)}{\chi(u+a)}.$$

8.

Let us, as above, put:

$$mK + m'iK' = n\omega,$$

while n denotes an odd number, m, m' arbitrary positive or negative numbers of such a kind that the numbers m, m', n are not divisible by the same number; from the preceding:

$$\chi(u + 4n\omega) = \chi(u).$$

Now let us form the product

$$\frac{\chi(u)\chi(u+4\omega)\chi(u+8\omega)\cdots\chi(u+4(n-1)\omega)}{\chi^2(4\omega)\chi^2(8\omega)\cdots\chi^2(2(n-1)\omega)} = \psi(u),$$

it is plain, having put $u + 4\omega$ instead of u, that each factor goes over into the following, but the last into the first; hence, since the product of all of them is not changed:

$$\psi(u + 4\omega) = \psi(\omega),$$

and hence even, while p denotes an arbitrary positive or negative number:

$$\psi(u + 4p\omega) = \psi(u).$$

Now, since in general:

$$\chi(u + 4(n-p)\omega) = \chi(u - 4p\omega),$$

from (14.):

$$\frac{\chi(u+4\omega)\chi(u+4(n-1)\omega)}{\chi^2(4\omega)} = (1-k^2\sin^2\operatorname{am}4\omega\sin^2\operatorname{am}u)\chi^2(u),$$
$$\frac{\chi(u+8\omega)\chi(u+4(n-2)\omega)}{\chi^2(8\omega)} = (1-k^2\sin^2\operatorname{am}8\omega\sin^2\operatorname{am}u)\chi^2(u),$$
$$\dots \qquad\qquad \dots$$

whence the product $\psi(u)$ can also be exhibited this way:

$$(16.)\qquad \psi(u) = \chi^n(u)[1+B'\sin^2\operatorname{am}u + B''\sin^4\operatorname{am}u + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}u],$$

if, as above, one puts the denominator of the substitution:

$$(1-k^2\sin^2\operatorname{am}4\omega\sin^2\operatorname{am}u)\cdots(1-k^2\sin^2\operatorname{am}2(n-1)\omega\sin^2\operatorname{am}u)$$
$$= 1+B'\sin^2\operatorname{am}u + B''\sin^4\operatorname{am}u + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}u.$$

Now, since $\psi(u+4p\omega) = \psi(u)$, from (16.) the following fundamental formula of highest importance follows

$$(17.)\qquad \frac{\chi(u+4p\omega)}{\chi(u)}$$
$$= \sqrt[n]{\frac{1+B'\sin^2\operatorname{am}u + B''\sin^4\operatorname{am}u + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}u}{1+B'\sin^2\operatorname{am}(u+4p\omega) + B''\sin^4\operatorname{am}(u+4p\omega) + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}(u+4p\omega)}}.$$

Having put $u=0$, from (17.):

$$(18.)\qquad \chi(4p\omega) = \frac{1}{\sqrt[n]{1+B'\sin^2\operatorname{am}4p\omega + B''\sin^4\operatorname{am}4p\omega + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}4p\omega}}.$$

9.

Having put $\sin\operatorname{am}u = x$, since:

$$\sin\operatorname{am}(u\pm a) = \frac{x\cos\operatorname{am}a\Delta\operatorname{am}\pm\sqrt{(1-xx)(1-k^2xx)}\sin\operatorname{am}a}{1-k^2\sin^2\operatorname{am}a\cdot xx},$$

we see that the expression

$$\frac{1+B'\sin^2\mathrm{am}(u\pm 4p\omega)+B''\sin^4\mathrm{am}(u\pm 4p\omega)+\cdots+B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\mathrm{am}(u\pm 4p\omega)}{1+B'\sin^2\mathrm{am}\,4p\omega+B''\sin^4\mathrm{am}\,4p\omega+\cdots+B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\mathrm{am}\,4p\omega}$$

takes on the form:

$$\frac{V(p)\pm\sqrt{(1-xx)(1-k^2xx)}W^{(p)}}{(1-k^2\sin^2\mathrm{am}\,4p\omega\cdot xx)^{n-1}},$$

where $V^{(p)}$, $W^{(p)}$ denote polynomial functions of x. Hence, if one additionally puts:

$$V=1+B'\sin^2\mathrm{am}\,u+B''\sin^4\mathrm{am}\,u+\cdots+B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\mathrm{am}\,u,$$

from (17.), (18.):

$$\text{(19.)}\qquad \frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}=\sqrt[n]{\frac{V(1-k^2\sin^2\mathrm{am}\,4p\omega\cdot xx)^{n-1}}{V^{(p)}+\sqrt{(1-xx)(1-k^2xx)}W^{(p)}}},$$

$$\text{(20.)}\qquad \frac{\chi(u-4p\omega)}{\chi(4p\omega)\chi(u)}=\sqrt[n]{\frac{V(1-k^2\sin^2\mathrm{am}\,4p\omega\cdot xx)^{n-1}}{V^{(p)}-\sqrt{(1-xx)(1-k^2xx)}W^{(p)}}}.$$

Having multiplied them by each other, since from (14.):

$$\frac{\chi(u+4p\omega)\chi(u-4p\omega)}{\chi^2(4p\omega)\chi^2(u)}=1-k^2\sin^2\mathrm{am}\,4p\omega\cdot xx,$$

we obtain:

$$[1-k^2\sin^2 4p\omega\cdot xx]^n=\frac{VV(1-k^2\sin^2\mathrm{am}\,4p\omega\cdot xx)^{2n-2}}{V^{(p)}V^{(p)}-(1-xx)(1-k^2xx)W^{(p)}W^{(p)}},$$

or:

$$V^{(p)}V^{(p)}-(1-xx)(1-k^2xx)W^{(p)}W^{(p)}=VV(1-k^2\sin^2\mathrm{am}\,4p\omega\cdot xx)^{n-2}.$$

The function V contains the factor $1-k^2\sin^2\mathrm{am}\,4p\omega\cdot xx$ so that, having put

$$V=V_p(1-k^2\sin^2\mathrm{am}\,4p\omega\cdot xx),$$

v_p is a polynomial function: Having substituted this, we find:

$$(21.) \qquad V^{(p)}V^{(p)} - (1-xx)(1-k^2xx)W^{(p)}W^{(p)} = V_pV_p(1-k^2\sin^2\operatorname{am}4p\omega\cdot xx)^n.$$

Hence from (19.), (20.) it easily follows:

$$(22.) \qquad \frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)} = \sqrt[n]{\frac{V^{(p)} - \sqrt{(1-xx)(1-k^2xx)}W^{(p)}}{V_p}},$$

$$(23.) \qquad \frac{\chi(u-4p\omega)}{\chi(4p\omega)\chi(u)} = \sqrt[n]{\frac{V^{(p)} + \sqrt{(1-xx)(1-k^2xx)}W^{(p)}}{V_p}}.$$

Additionally,	$V^{(p)}$	will be a function of x			of even order	$2n-4$,
	$W^{(p)}$	$--$	$--$	$--$	of odd order	$2n-5$,
	V_p	$--$	$--$	$--$	of even order	$n-3$.

10.

For the sake of brevity, let us put:

$$\Phi(u) = 1 + B'\sin^2\operatorname{am}u + B''\sin^4\operatorname{am}u + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}u,$$

from (17.):

$$(24.) \qquad \frac{\chi(u+4p\omega)}{\chi(u)} = \sqrt[n]{\frac{\Phi(u)}{\Phi(u+4p\omega)}};$$

after logarithmic differentiation it results:

$$(25.) \qquad \frac{\chi'(u+4p\omega)}{\chi(u+4p\omega)} - \frac{\chi'(u)}{\chi(u)} = \frac{1}{n}\frac{\Phi'(u)}{\Phi(u)} - \frac{1}{n}\frac{\Phi'(u+4p\omega)}{\Phi(u+4p\omega)},$$

if one puts

$$\chi'(u) = \frac{d\chi(u)}{du}, \quad \phi'(u) = \frac{d\Phi(u)}{du}.$$

Further, since $\chi(u) = e^{ruu}\Omega(u)$:

$$\frac{\chi'(u)}{\chi(u)} = 2ru + \frac{\Omega'(u)}{\Omega(u)} = 2ru + E(u),$$

if $\Omega'(u) = \dfrac{d\Omega(u)}{du}$. Having put $u = 0$, from (25.) you will find:

$$\frac{\chi'(4p\omega)}{\chi(4p\omega)} = -\frac{1}{n}\frac{\Phi'(4p\omega)}{\Phi(4p\omega)},$$

or, for the sake of brevity having put $\operatorname{am} 4p\omega = \alpha_p$:

$$(26.) \qquad E(4p\omega) + 8rp\omega = -\frac{1}{n}\frac{\Phi'(4p\omega)}{\Phi(4p\omega)}$$

$$= -\frac{1}{n}\frac{\cos\alpha_p \Delta\alpha_p [2B' \sin\alpha_p + 4B'' \sin^3\alpha_p + \cdots + (n-1)B^{\left(\frac{n-1}{2}\right)} \sin^{n-2}\alpha_2]}{1 + B' \sin^2\alpha_p + B'' \sin^4\alpha_p + \cdots + B^{\left(\frac{n-1}{2}\right)} \sin^{n-1}\alpha_p}.$$

This formula teaches, how the elliptic integrals of the second kind can be exhibited in the case in which the argument is a certain part of $4(mK + m'iK')$.

From formula (15.) we obtain:

$$\Pi(u,a) = u\frac{\chi'(a)}{\chi(a)} + \frac{1}{2}\log\frac{\chi(u-a)}{\chi(u+a)} = u[E(a) + 2ra] + \frac{1}{2}\log\frac{\chi(u-a)}{\chi(u+a)},$$

whence, recalling (24.):

$$\begin{aligned}
(27.) \qquad \Pi(4p\omega, a) &= 4p\omega[E(a) + 2ra] + \tfrac{1}{2}\log\frac{\chi(a - 4p\omega)}{\chi(4p\omega))} \\
&= 4p\omega[E(a) + 2ra] + \frac{1}{2n}\log\frac{\Phi(a - 4p\omega)}{\Phi(a - 4p\omega))} \\
(28.) \qquad \Pi(u, 4p\omega) &= u[E(4p\omega) + 2rp\omega] + \tfrac{1}{2}\log\frac{\chi(u - 4p\omega)}{\chi(u + 4p\omega))} \\
&= -\frac{u}{n}\cdot\frac{\Phi'(4p\omega)}{\Phi(4p\omega)} + \frac{1}{2n}\log\frac{\Phi(u + 4p\omega)}{\Phi(u - 4p\omega)},
\end{aligned}$$

or:

$$(29.) \qquad \Pi(4p\omega, a) = 4p\omega[E(a) + 2ra]$$

$$+\frac{1}{2n}\log\frac{1 + B'\sin^2\operatorname{am}(a + 4p\omega) + B''\sin^4\operatorname{am}(a + 4p\omega) + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}(a + 4p\omega)}{1 + B'\sin^2\operatorname{am}(a - 4p\omega) + B''\sin^4\operatorname{am}(a - 4p\omega) + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\operatorname{am}(a - 4p\omega)},$$

$$(30.)\qquad \Pi(u, 4p\omega)$$
$$= -\frac{u}{n}\cdot\frac{\cos\alpha_p\Delta_p[2B'\sin\alpha_p + 4B''\sin^2\alpha_p + \cdots + (n-1)B^{\left(\frac{n-1}{2}\right)}\sin^{n-2}\alpha]}{1 + B'\sin^2\alpha_a + B''\sin^4\alpha_a + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\alpha}$$
$$+\frac{1}{2n}\log\frac{1 + B'\sin^2\mathrm{am}(a+4p\omega) + B''\sin^4\mathrm{am}(a+4p\omega) + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\mathrm{am}(a+4p\omega)}{1 + B'\sin^2\mathrm{am}(a-4p\omega) + B''\sin^4\mathrm{am}(a-4p\omega) + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\mathrm{am}(a-4p\omega)}.$$

These formulas tell us, how the elliptic integrals of the third kind can be exhibited in the cases in which the argument of either the amplitude or the parameter (see *Fund.* § 49) is a certain part of $4(mK + m'iK')$.

11.

In the fundamental formula (24.):

$$\frac{\chi(u+4p\omega)}{\chi(u)} = \sqrt[n]{\frac{\Phi(u)}{\Phi(u+4p\omega)}}$$
$$= \sqrt[n]{\frac{1 + B'\sin^2\mathrm{am}\,u + B''\sin^4\mathrm{am}\,u + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\mathrm{am}\,u}{1 + B'\sin^2\mathrm{am}(u+4p\omega) + B''\sin^4\mathrm{am}(u+4p\omega) + \cdots + B^{\left(\frac{n-1}{2}\right)}\sin^{n-1}\mathrm{am}(u+4p\omega)}}$$

the one side of the equation contains the function $\chi(u)$, which has one period, but the other side consists of the function $\sin\mathrm{am}\,u$, which, except for this one period, enjoys another one and is hence double-periodic. Therefore, while you apply the period to the other side, the expression

$$\frac{\chi(u+4p\omega)}{\chi(u)}$$

will certainly be changed, but cannot undergo another change than the one resulting from the ambiguity of the n-th root. Now let us prove this very deep theorem that the expression

$$\frac{\chi(u+4p\omega)}{\chi(u)},$$

which has one period with the elliptic functions in common, while you apply another period, which those enjoy, does not undergo another change than that it is multiplied by the root of the equation $x^n = 1$, from the nature of the

function $\chi(u)$.
Let us put

$$mK + M'iK' = Q, \qquad \mu K + \mu' iK' = Q',$$

further let:

$$aK + a'iK' = pQ + P'Q',$$

whence, if p, p', m, m', μ, μ' are real quantities:

$$a = pm + P'\mu, \qquad a' = pM' + p'\mu'$$

and hence:

$$p = \frac{\mu' a - \mu a'}{m\mu' - m'\mu}, \qquad p' = \frac{ma' - m'a}{m\mu' - m'\mu}.$$

Let m, m', μ, μ' be arbitrary positive or negative numbers of such a kind that

$$m\mu' - m'\mu = 1,$$

it will be:

$$p = \mu' a - \mu a', \quad p' = ma' - m'a,$$

whence it is plain, whatever integer numbers a, a' are, that also p, p' are also integers and vice versa. Further:

$$K = \mu' Q - m'Q', \qquad iK' = mQ' - \mu Q.$$

Whatever positive or negative integer numbers a, a' are, it will be

$$\sin\operatorname{am}(u + 4aK + 4a'iK') = \sin\operatorname{am} u,$$

whence, whatever positive or negative integer numbers p, p' are:

$$\sin\operatorname{am}(u + 4pQ + 4p'Q') = \sin\operatorname{am} u.$$

It is possible to compose all innumerable periods, which the elliptic functions enjoy, from the two contained in the equations:

$$\sin\operatorname{am}(u + 4K) = \sin\operatorname{am} u, \qquad \sin\operatorname{am}(u + 4iK') = \sin\operatorname{am} u.$$

From the preceding it is plain that one can substitute these for those:

$$\sin\operatorname{am}(u+4Q)=\sin\operatorname{am}u,\qquad \sin\operatorname{am}(u+4Q')=\sin\operatorname{am}u,$$

if:

$$Q=mK+m'iK',\quad Q'=uK+\mu'iK',$$

while m, m', μ, μ' denote positive or negative integers of such a kind that $m\mu'-m'\mu=1$. Hence we see that the periods, which the elliptic functions enjoy, can be composed from two in innumerable ways. But we will call two periods of such a kind, from which all remaining one can be composed, CONJUGATED periods.

Having, as above, put $\omega=\dfrac{Q}{n}=\dfrac{mK+m'iK'}{n}$, let us now ask, what is propounded, what becomes out of the expression

$$\frac{\chi(u+4p\omega)}{\chi(u)}=\frac{\chi\left(u+\frac{4pQ}{n}\right)}{\chi(u)},$$

having changed u into $u+4Q'$ or more generally into $u+4P'Q'$, while p' denotes an arbitrary positive or negative number. We saw, having put:

$$r=\frac{m'i\pi}{4K(mK+m'iK')}-\frac{E}{2K}=\frac{m'i\pi}{4KQ}-\frac{E}{2K},$$

it will be:

$$e^{r(u+4Q)^2}\Omega(u+4Q)=e^{ruu}\Omega(u);$$

hence, having put μ, μ' instead of m, m', and hence Q' instead of Q, and

$$r'=\frac{\mu'i\pi}{4KQ'}-\frac{E}{2K},$$

we find:

$$e^{r'(u+4Q')^2}\Omega(u+4Q')=e^{r'uu}\Omega(u).$$

Having changed u into $u+\dfrac{4pQ}{n}=u+4p\omega$, it results:

$$e^{r'\left(u+\frac{4pQ}{n}+4Q'\right)^2}\Omega\left(u+\frac{4pQ}{n}+4Q'\right)=e^{r'\left(u+\frac{4pQ}{n}\right)^2}\Omega(u+\frac{4pQ}{n}),$$

whence:

$$e^{r' \frac{32pQQ'}{n}} \cdot \frac{\Omega\left(u + \frac{4pQ}{n} + 4Q'\right)}{\Omega(u + 4Q')} = \frac{\Omega\left(\frac{4pQ}{n}\right)}{\Omega(u)}.$$

But from the formula $\chi(u) = e^{ruu}\Omega(u)$ it follows:

$$\begin{aligned}\frac{\chi\left(u + \frac{4pQ}{n} + 4Q'\right)}{\chi(u + 4Q')} &= e^{r \cdot \frac{4pQ}{n}\left(2u + 9Q' + \frac{4pQ}{n}\right)} \cdot \frac{\Omega\left(u + \frac{4pQ}{n} + 4Q'\right)}{\Omega(u + 4Q')} \\ &= e^{r \cdot \frac{4pQ}{n}\left(2u + 8Q' + \frac{4pQ}{n}\right) - r' \cdot \frac{32pQQ'}{n}} \cdot \frac{\Omega\left(u + \frac{4pQ}{n}\right)}{\Omega(u)},\end{aligned}$$

whence:

$$\frac{\chi\left(u + \frac{4pQ}{n} + 4Q'\right)}{\chi(u + 4Q')} = e^{\frac{32pQQ'}{n}(r - r')} \cdot \frac{\chi\left(u + \frac{4pQ}{n}\right)}{\chi(u)}.$$

But

$$r - r' = \frac{m'i\pi}{4KQ} - \frac{\mu' i\pi}{4KQ} = \frac{i\pi}{K} \cdot \frac{m'Q' - \mu' Q}{4QQ'}$$

and hence, since $m'Q' - \mu'Q = -K$:

$$\frac{32pQQ'}{n}(r - r') = -\frac{8ip\pi}{n},$$

whence we obtain the fundamental formula:

$$\text{(31.)} \qquad \frac{\chi\left(u + \frac{4pQ}{n} + 4Q'\right)}{\chi(u + 4Q')} = e^{-\frac{8ip\pi}{n}} \cdot \frac{\left(u + \frac{4pQ}{n}\right)}{\chi(u)}$$

or this more general one:

$$\text{(32.)} \qquad \frac{\chi\left(u + \frac{4pQ}{n} + 4p'Q'\right)}{\chi(u + 4p'Q')} = e^{-\frac{8ipp'\pi}{n}} \cdot \frac{\left(u + \frac{4pQ}{n}\right)}{\chi(u)}$$

Therefore, we see, what was to be demonstrated, THAT THE EXPRESSION

$$\frac{\chi\left(u + \frac{4pQ}{n}\right)}{\chi(u)} = \frac{\chi(u + 4p\omega)}{\chi(u)},$$

which has one period with the elliptic function in common or remains unchanged, having changed u into $u + 4Q$, while you apply the conjugated period to it or u is changed into $u + 4Q'$, is multiplied by the $n-$th root of 1.
Yes, formula (32.) even suggests an analytical expression of the $n-$th root of 1 you have to find; this is a beautiful result.
These considerations now pave the way to greater results. For, on these as a foundation in the following we will discuss the inverse transformations and the section of elliptic functions, an intricate and elegant question.

Written in Königsberg, in the month of April 1829.

2 Second Treatise on elliptic functions

2.1 On the Sums of Series of elliptic Functions, whose Arguments constitute an arithmetic Series.

In the following we propound certain elementary formulas for the sums of elliptic functions, whose arguments constitute an arithmetic series. They both can be useful in other questions and yield general functions for the transformation of elliptic functions very easily.
I start from the known formula for the addition of elliptic integrals of the second kind:

$$(1.)\qquad E(a) + E(u) - E(a+u) = k^2 \sin\operatorname{am} a \sin\operatorname{am} u \sin\operatorname{am}(u+a),$$

in which from the notation introduced in the first treatise on elliptic functions:

$$E(u) = \int\limits_0^u \Delta^2 \operatorname{am} u du.$$

In formula (1.) let us write pa instead of a, whence:

$$E(pa) + E(u) - E(u+pa) = k^2 \sin\operatorname{am} pa \sin\operatorname{am} u \sin\operatorname{am}(u+pa);$$

and having successively put u, $u+a$, $u+2a$, $\cdots$, $u+(n-1)a$ instead of u, let us do the summation. Generally denoting the sum by $\sum^{(n)} F(u)$:

$$\sum^{(n)} F(u) = F(u) + F(u+a) + F(u+2a) + \cdots + F(u+(n-1)a),$$

we have:

$$nE(pa) + \sum^{(n)} E(u) - E^{(n)} E(u+pa) = k^2 \sin\operatorname{am} pa \sum^{(n)} \sin\operatorname{am}(u+pa).$$

The same way, from the formula:

$$E(na) + E(u) - E(u+na) = k^2 \sin\operatorname{am} na \sin\operatorname{am} u \sin\operatorname{am}(u+na),$$

having successively put u, $u+a$, $u+2a$, $\cdots$, $u+(p-1)a$ instead of u and having done the substitution, you obtain:

$$pE(na)+\sum^{(p)}E(u)-\sum(p)E(u+na)=k^2\sin\mathrm{am}\,na\sum^{(n)}\sin\mathrm{am}\,u\sin\mathrm{am}(u+na).$$

Now I observe that:

$$\sum^{(n+p)}E(u)=\sum^{(n)}E(u)+\sum^{(p)}E(u+na)=\sum^{(p)}E(u)+\sum^{(n)}E(u+pa)$$

and hence:

$$\sum^{(n)}E(u)-\sum^{(n)}E(u+pa)=\sum^{(p)}E(u)-\sum^{(p)}E(u+na).$$

Hence from the mentioned two formulas

$$(2.)\qquad \left\{\begin{array}{r} k^2\sin\mathrm{am}\,pa\sum^{(n)}\sin\mathrm{am}\,u\sin\mathrm{am}(u+pa) \\ -k^2\sin\mathrm{am}\,na\sum^{(p)}\sin\mathrm{am}\,u\sin\mathrm{am}(u+na) \end{array}\right\}=nE(pa)-pE(na).$$

We have the memorable case in which $\sin\mathrm{am}\,na$ but not $\sin\mathrm{am}\,pa$ do not vanish at the time, in which case (2.):

$$(3.)\qquad \sum^{(n)}\sin\mathrm{am}\,u\sin\mathrm{am}(u+pa)=\frac{nE(pa)-pE(na)}{k^2\sin\mathrm{am}\,pa}.$$

Now I observe that in the elements one proves the formulas:

$$\begin{aligned}\cos\mathrm{am}\,a &= \cos\mathrm{am}\,u\,\cos\mathrm{am}(u+a)+\quad\Delta\,\mathrm{am}\,a\sin\mathrm{am}\,u\sin\mathrm{am}(u+a),\\ \Delta\,\mathrm{am}\,a &= \quad\Delta\,\mathrm{am}\,u\quad\Delta\,\mathrm{am}(u+a)+k^2\cos\mathrm{am}\,a\sin\mathrm{am}\,u\sin\mathrm{am}(u+a),\end{aligned}$$

whence from (3.) we also obtain:

$$(4.)\qquad \sum^{(n)}\cos\mathrm{am}\,u\,\cos\mathrm{am}(u+pa)=n\cos\mathrm{am}\,pa-\frac{\Delta\,\mathrm{am}\,pa}{k^2\sin\mathrm{am}\,pa}[nE(pa)-pE(na)],$$

$$(5.)\qquad \sum^{(n)}\quad\Delta\,\mathrm{am}\,u\quad\Delta\,\mathrm{am}(u+a)=\quad n\Delta\,\mathrm{am}\,pa-\quad\cot\mathrm{am}\,pa[nE(pa)-pE(na)].$$

Therefore, we see, if $\sin\operatorname{am} na$ vanishes, but $\sin\operatorname{am} pa$ does not vanish at the same time, that the expressions

$$\textstyle\sum^{(n)} \ \sin\operatorname{am} u \ \sin\operatorname{am}(u+pa),$$

$$\textstyle\sum^{(n)} \ \cos\operatorname{am} u \ \cos\operatorname{am}(u+pa),$$

$$\textstyle\sum^{(n)} \quad \Delta\operatorname{am} u \qquad \Delta(u+pa)$$

do not depend on the argument u. Furthermore, having, as in the *Fundamenta nova*, put

$$\omega = \frac{mK + m'iK'}{n},$$

while m, m' denote arbitrary positive or negative numbers, which do not have the same factor of n in common, so that $\sin\operatorname{am} na$ and $\sin\operatorname{am} pa$ do not vanish at the same time, it has to be $a = 2\mu\omega$, while μ denotes an arbitrary integer number, as long as μp is not divisible by n.
Not using the sums of the elliptic functions you obtain the formulas this way. For, having put:

$$\operatorname{am} u = \alpha, \qquad \operatorname{am} v = \beta, \qquad \operatorname{am}(u+v) = \sigma, \qquad \operatorname{am}(u-v) = \vartheta,$$

from the formulas (24.) – (29.) in the *Fundam.* § 18 it follows:

$$\begin{aligned}
\cos\sigma \quad \Delta\vartheta \ + \cos\vartheta \quad \Delta\sigma &= \frac{2\cos\beta\Delta\beta\cos\alpha\Delta\alpha}{1-k^2\sin^2\beta\sin^2\alpha},\\
\Delta\sigma \ \sin\vartheta \ + \quad \Delta\vartheta \ \sin\sigma &= \frac{2\cos\beta\sin\alpha\Delta\alpha}{1-k^2\sin^2\beta\sin^2\alpha},\\
\sin\sigma\cos\vartheta \ + \ \sin\vartheta\cos\sigma &= \frac{2\Delta\beta\sin\alpha\cos\alpha}{1-k^2\sin^2\beta\sin^2\alpha}.
\end{aligned}$$

But at the same time we gave the formulas (4.) – (6.) in § 18:

$$\sin\sigma - \sin\vartheta = \frac{2\sin\beta\cos\alpha\Delta\alpha}{1-k^2\sin^2\beta\sin^2\alpha},$$
$$\cos\vartheta - \cos\sigma = \frac{2\sin\beta\Delta\beta\sin\alpha\Delta\alpha}{1-k^2\sin^2\beta\sin^2\alpha},$$
$$\Delta\vartheta - \Delta\sigma = \frac{2k^2\sin\beta\cos\beta\sin\alpha\cos\alpha}{1-k^2\sin^2\beta\sin^2\alpha},$$

having combined which with the first it results:

$$(6.)\qquad \cos\sigma\ \ \Delta\vartheta + \cos\vartheta\ \ \Delta\sigma = \frac{\Delta\beta}{\tan\beta}(\sin\sigma - \sin\vartheta),$$
$$(7.)\qquad \Delta\sigma\ \sin\vartheta + \ \Delta\vartheta\ \sin\sigma = \frac{1}{\Delta\beta\tan\beta}(\cos\vartheta - \cos\sigma),$$
$$(8.)\qquad \sin\sigma\cos\vartheta + \sin\vartheta\cos\sigma = \frac{\Delta\beta}{k^2\sin\beta\cos\beta}(\Delta\vartheta - \Delta\sigma).$$

Having put $u+\frac{a}{2}$ instead u and $v=\frac{a}{2}$:

$$\beta = \mathrm{am}\,\frac{a}{2},\quad \sigma = \mathrm{am}(u+a),\quad \vartheta = \mathrm{am}\,u,$$

whence (6.) – (8.) are represented this way:

$$\cos\mathrm{am}\,u\ \ \Delta\,\mathrm{am}(u+a) + \ \cos\mathrm{am}(u+a)\Delta\,\mathrm{am}\,u = \frac{\Delta\,\mathrm{am}\,\frac{a}{2}}{\tan\mathrm{am}\,\frac{a}{2}}[\sin\mathrm{am}(u+a) - \sin\mathrm{am}\,u],$$
$$\Delta\,\mathrm{am}\,u\ \sin\mathrm{am}(u+a) + \ \Delta\,\mathrm{am}(u+a)\sin\mathrm{am}\,u = \frac{1}{\Delta\,\mathrm{am}\,\frac{a}{2}\tan\mathrm{am}\,\frac{a}{2}}[\cos\mathrm{am}\,u - \cos\mathrm{am}(u+a)],$$
$$\sin\mathrm{am}\,u\cos\mathrm{am}(u+a) + \sin\mathrm{am}(u+a)\cos\mathrm{am}\,u = \frac{\Delta\,\mathrm{am}\,\frac{a}{2}}{k^2\sin\mathrm{am}\,\frac{a}{2}\cos\mathrm{am}\,\frac{a}{2}}[\Delta\,\mathrm{am}\,u - \Delta\,\mathrm{am}(u+a)].$$

In these formulas write pa instead of a, and having successively put u, $u+a$, $\cdots$, $u+(n-1)a$ instead of u, do the summation; further, in these same formulas write na instead of a, and having successively put u, $u+a$, $\cdots$, $u+(p-1)a$, do the summation again. Having compared both sums to each other and having additionally observed that:

$$\sum^{(n)} F(u) - \sum^{(n)} F(u+pa) = \sum^{(p)} F(u) - \sum^{(p)} F(u+na),$$

you obtain:

$$(9.)\qquad \frac{\tan\mathrm{am}\,\frac{pa}{2}}{\Delta\,\mathrm{am}\,\frac{pa}{2}}\,\Sigma^{(n)}[\cos\mathrm{am}\,u\Delta\,\mathrm{am}(u+pa)+\cos\mathrm{am}(u+pa)\Delta\,\mathrm{am}\,u]$$

$$=\frac{\tan\mathrm{am}\,\frac{na}{2}}{\Delta\,\mathrm{am}\,\frac{na}{2}}\,\Sigma^{(p)}[\cos\mathrm{am}\,u\Delta\,\mathrm{am}(u+na)+\cos\mathrm{am}(u+na)\Delta\,\mathrm{am}\,u],$$

$$(10.)\qquad \tan\frac{pa}{2}\Delta\,\mathrm{am}\,\frac{pa}{2}\,\Sigma^{(n)}[\Delta\,\mathrm{am}\,u\sin\mathrm{am}(u+pa)+\Delta\,\mathrm{am}(u+pa)\sin\mathrm{am}\,u]$$

$$=\tan\mathrm{am}\,\frac{na}{2}\Delta\,\mathrm{am}\,\tfrac{na}{2}\,\Sigma^{(p)}[\Delta\,\mathrm{am}\,u\sin\mathrm{am}(u+na)+\Delta\,\mathrm{am}(u+na)\sin\mathrm{am}\,u],$$

$$(11.)\qquad \frac{\sin\mathrm{am}\,\frac{pa}{2}\cos\mathrm{am}\,\frac{pa}{2}}{\Delta\,\mathrm{am}\,\frac{pa}{2}}\,\Sigma^{(n)}[\sin\mathrm{am}\,u\cos\mathrm{am}(u+pa)+\sin\mathrm{am}(u+pa)\cos\mathrm{am}\,u]$$

$$=\frac{\sin\mathrm{am}\,\frac{na}{2}\cos\mathrm{am}\,\frac{na}{2}}{\Delta\,\mathrm{am}\,\frac{na}{2}}\,\Sigma^{(p)}[\sin\mathrm{am}\,u\cos\mathrm{am}\,/u+na)+\sin\mathrm{am}(u+na)\cos\mathrm{am}\,u].$$

In the special case in which $\sin\mathrm{am}\,\frac{na}{2}$ and $\sin\mathrm{am}\,pa$ do not vanish at the same time, from (9.) $-$ (11.) these memorable formulas follow:

$$(12.)\qquad \Sigma^{(n)}[\cos\mathrm{am}\,u\quad\Delta\,\mathrm{am}(u+pa)\;+\;\cos\mathrm{am}(u+pa)\quad\Delta\,\mathrm{am}\,u]=0,$$

$$(13.)\qquad \Sigma^{(n)}[\;\;\Delta\,\mathrm{am}\,u\;\;\sin\mathrm{am}(u+pa)\;+\;\;\Delta\,\mathrm{am}(u+pa)\;\;\sin\mathrm{am}\,u]=0,$$

$$(14.)\qquad \Sigma^{(n)}[\sin\mathrm{am}\,u\;\cos\mathrm{am}(u+pa)\;+\;\sin\mathrm{am}(u+pa)\;\cos\mathrm{am}\,u]=0.$$

Now we constructed general formulas for the transformation of elliptic functions by means of the formulas (3.) $-$ (5.), (12.) $-$ (14.).

2.2 New Proof of the fundamental formulas for the transformation of elliptic functions

Let us consider the expressions:

$$R = \sin\operatorname{am} u + \sin\operatorname{am}(u+4\omega) + \sin\operatorname{am}(u+8\omega) + \cdots + \sin\operatorname{am}(u+4(n-1)\omega),$$

$$S = \cos\operatorname{am} u + \cos\operatorname{am}(u+4\omega) + \cos\operatorname{am}(u+8\omega) + \cdots + \cos\operatorname{am}(u+4(n-1)\omega),$$

$$T = \Delta\operatorname{am} u + \Delta\operatorname{am}(u+4\omega) + \Delta\operatorname{am}(u+8\omega) + \cdots + \Delta\operatorname{am}(u+4(n-1)\omega),$$

in which n is an odd number, $\omega = \frac{mK+m'iK'}{n}$, as above and in the *Fundamenta nova*, so that, having put $4\omega = a$, if $p < n$ or certainly p is not divisible by n, $\sin\operatorname{am}\frac{na}{2} = 0$ but not $\sin pa = 0$ at the same time.
Here, for the sake of brevity, by $\sum F(u)$ we denote the sum:

$$\sum F(u) = F(u) + F(u+4\omega) + \cdots + F(u+4(n-1)\omega),$$

the expressions R, S, T can be represented in shorter form this way:

$$R = \sum \sin\operatorname{am} u, \qquad S = \sum \cos\operatorname{am} u, \qquad T = \sum \Delta\operatorname{am} u.$$

Let us ask for the squares and the products of two of the expressions R, S, T. As it is plain having done the multiplication, we have:

$$\begin{aligned}
RR = \textstyle\sum \sin^2\operatorname{am} u &+ \textstyle\sum \sin\operatorname{am} u \ \sin\operatorname{am}(u+4\omega) \\
&+ \textstyle\sum \sin\operatorname{am} u \ \sin\operatorname{am}(u+8\omega) \\
&+ \quad \cdots \qquad \cdots \\
&+ \textstyle\sum \sin\operatorname{am} u \ \sin\operatorname{am}(u+4(n-1)\omega), \\
SS = \textstyle\sum \cos^2\operatorname{am} u &+ \textstyle\sum \cos\operatorname{am} u \ \cos\operatorname{am}(u+4\omega) \\
&+ \textstyle\sum \cos\operatorname{am} u \ \cos\operatorname{am}(u+8\omega) \\
&+ \quad \cdots \qquad \cdots \\
&+ \textstyle\sum \cos\operatorname{am} u \ \cos\operatorname{am}(u+4(n-1)\omega), \\
TT = \textstyle\sum \Delta^2\operatorname{am} u &+ \textstyle\sum \Delta\operatorname{am} u \quad \Delta\operatorname{am}(u+4\omega) \\
&+ \textstyle\sum \Delta\operatorname{am} u \quad \Delta\operatorname{am}(u+8\omega) \\
&+ \quad \cdots \qquad \cdots \\
&+ \textstyle\sum \Delta\operatorname{am} u \quad \Delta\operatorname{am}(u+4(n-1)\omega).
\end{aligned}$$

Now from the results we propounded above it is plain that expressions of this kind:

$$\sum \sin\operatorname{am} u \ \sin\operatorname{am}(u+4p\omega),$$
$$\sum \cos\operatorname{am} u \ \cos\operatorname{am}(u+4p\omega),$$
$$\sum \ \Delta\operatorname{am} u \qquad \Delta(u+4p\omega),$$

in which as in the preceding $p < n$, are equal to constants or do not depend on the argument u. Hence one can put:

$$\text{(15.)} \qquad \left\{ \begin{aligned} RR &= \sum \sin^2\operatorname{am} u - 2\rho, \\ SS &= \sum \cos^2\operatorname{am} u - 2\sigma, \\ TT &= \sum \ \Delta^2\operatorname{am} u - 2\tau, \end{aligned} \right\}$$

while ρ, σ, τ denote constants, whose values must be taken from special values of u. For this aim, note the following elementary formulas:

$$\begin{aligned} \sin\operatorname{am} 4(n-n')\omega &= -\sin\operatorname{am} 4n'\omega, \\ \cos\operatorname{am}(K+4(n-n')\omega) &= = \cos\operatorname{am}(K+4n'\omega), \\ \Delta\operatorname{am}(K+iK'+4(n-n')\omega) &= - \ \Delta\operatorname{am}(K+iK'+4n'\omega), \end{aligned}$$

further, the formulas

$$\sin\operatorname{am} 0 = \cos\operatorname{am} K = \Delta\operatorname{am}(K+iK') = 0,$$

from which it is clear, having respectively put $u = 0$, $u = K$, $u = K + iK'$, that the expressions R, S, T and hence also RR, SS, TT vanish. Hence, since furthermore:

$$\Delta\operatorname{am}(K+iK'+u) = ik' \tan\operatorname{am} u,$$

from (15.), having respectively put $u = 0$, $u = K$, $u = K + iK'$ put:

$$\begin{aligned} \rho &= \sin^2\operatorname{am} 4\omega + \sin^2\operatorname{am} 8\omega + \cdots + \sin^2\operatorname{am} 2(n-1)\omega, \\ \sigma &= \cos^2\operatorname{coam} 4\omega + \cos^2\operatorname{coam} 8\omega + \cdots + \cos^2\operatorname{coam} 2(n-1)\omega, \\ \tau &= k'k'[\tan^2\operatorname{am} 4\omega + \tan^2\operatorname{am} 8\omega + \cdots + \tan^2\operatorname{am} 2(n-1)\omega]. \end{aligned}$$

The quantities ρ, σ, τ are the same as those we exhibited in the first treatise on elliptic functions denoted by these letters.
From the formulas (15.) it follows:

$$\begin{aligned} RR + SS &= n - \quad 2\rho - 2\sigma, \\ k^2 RR + TT &= n - 2k^2\rho + 2\tau, \end{aligned}$$

whence one can put:

$$\begin{aligned} R &= \sqrt{n - 2\rho - 2\sigma} \cdot \sin\psi, \\ S &= \sqrt{n - 2\rho - 2\sigma} \cdot \cos\psi, \\ T &= \sqrt{n - 2k^2\rho + 2\tau} \cdot \sqrt{1 - \frac{k^2(n - 2\rho - 2\sigma)}{n - 2k^2\rho + 2\tau}\sin^2\psi}, \end{aligned}$$

or, having put:

$$\frac{k^2(n - 2\rho - 2\sigma)}{n - 2k^2\rho + 2\tau} = \lambda\lambda, \qquad n - 2k^2\rho + 2\tau = \frac{1}{MM},$$

we find:

$$r = \frac{\lambda}{kM}\sin\psi, \qquad S = \frac{\lambda}{kM}\cos\psi, \qquad T = \frac{1}{M}\sqrt{1 - \lambda\lambda\sin^2\psi}.$$

Let us now ask for the products of two of the expressions R, S, T. After the multiplication one finds:

$$\begin{aligned} ST = &\quad \textstyle\sum \cos\operatorname{am} u \Delta\operatorname{am} u \\ &+\tfrac{1}{2}\textstyle\sum[\cos\operatorname{am} u \Delta\operatorname{am}(u + 4\omega) + \cos\operatorname{am}(u + 4\omega)\Delta\operatorname{am} u] \\ &+\tfrac{1}{2}\textstyle\sum[\cos\operatorname{am} u \Delta\operatorname{am}(u + 8\omega) + \cos\operatorname{am}(u + 8\omega)\Delta\operatorname{am} u] \\ &+ \quad \cdots\cdots \\ &+\tfrac{1}{2}\textstyle\sum[\cos\operatorname{am} u \Delta\operatorname{am}(u + 4(n-1)\omega) + \cos\operatorname{am}(u + 4(n-1)\omega)\Delta\operatorname{am} u]. \end{aligned}$$

We added the factor $\frac{1}{2}$, since in the sums, to which it was added, each term occurs twice. Now from (12.), having put $a = 4\omega$, if, as in the preceding, $p < n$, we have:

$$\sum[\cos\operatorname{am} u \Delta\operatorname{am}(u + 4p\omega) + \cos\operatorname{am}(u + 4p\omega)\Delta\operatorname{am} u] = 0,$$

whence simply:

$$ST = \sum \cos \mathrm{am}\, u \Delta\, \mathrm{am}\, u.$$

The same way by means of formulas (13.), (14.) one finds:

$$TR = \sum\ \Delta\, \mathrm{am}\, u\ \sin \mathrm{am}\, u,$$
$$RS = \sum \sin \mathrm{am}\, u\ \cos \mathrm{am}\, u.$$

But from the formulas:

$$R = \sum \sin \mathrm{am}\, u, \qquad S = \sum \cos \mathrm{am}\, u, \qquad T = \sum \Delta\, \mathrm{am}\, u,$$

after a differentiation it follows:

$$\begin{aligned}
\frac{dR}{du} &= \qquad \sum \cos \mathrm{am}\, u \quad \Delta\, \mathrm{am}\, u = \quad ST, \\
\frac{dS}{du} &= - \quad \sum \quad \Delta\, \mathrm{am}\, u\ \sin \mathrm{am}\, u = \quad TR, \\
\frac{dT}{du} &= -k^2 \sum\ \sin \mathrm{am}\, u\ \cos \mathrm{am}\, u = -k^2 ST
\end{aligned}$$

whence, since from the preceding:

$$R = \frac{\lambda}{kM} \sin \psi, \quad S = \frac{\lambda}{kM} \cos \psi, \quad T = \frac{1}{M}\sqrt{1 - \lambda\lambda \sin^2 \psi},$$

we find:

$$\frac{d\psi}{du} = \frac{1}{M}\sqrt{1 - \lambda\lambda \sin^2 \psi} \quad \text{or} \quad \frac{du}{M} = \frac{d\psi}{\sqrt{1 - \lambda\lambda \sin^2 \psi}},$$

whence, since ψ and u vanish at the same time:

$$\psi = \mathrm{am}\left(\frac{u}{M}, \lambda\right).$$

Therefore, we obtained the values of R, S, T:

$$\begin{aligned}
R &= \frac{\lambda}{kM}\ \sin \mathrm{am}\left(\frac{u}{M}, \lambda\right), \\
S &= \frac{\lambda}{kM} \cos \mathrm{am}\left(\frac{u}{M}, \lambda\right), \\
T &= \frac{1}{M} \quad \Delta\, \mathrm{am}\left(\frac{u}{M}, \lambda\right),
\end{aligned}$$

or, what is the same:

$$
\begin{aligned}
\frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \sin\operatorname{am}u + \sin\operatorname{am}(u+4\omega) + \cdots + \sin\operatorname{am}(u+4(n-1)\omega),\\
\frac{\lambda}{kM}\cos\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \cos\operatorname{am}u + \cos\operatorname{am}(u+4\omega) + \cdots + \cos\operatorname{am}(u+4(n-1)\omega),\\
\frac{1}{M}\quad\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \Delta\operatorname{am}u + \Delta\operatorname{am}(u+4\omega) + \cdots + \Delta\operatorname{am}(u+4(n-1)\omega).
\end{aligned}
$$

These are the fundamental formulas for the transformation of elliptic functions.